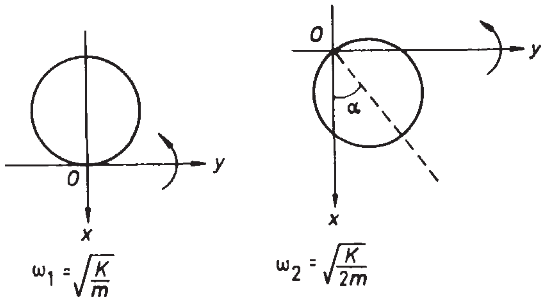

Fig. 1.60.

At the lower resonance, $r = A\cos(\omega t - \alpha)$, we have $\dot{\omega} = 0$, $\dot{r} = -\omega A\sin(\omega t - \alpha)$, so that

$$F = m[-2A\omega^2 \sin(\omega t - \alpha) + g\sin(\omega t)] ,$$

giving the torque as

$$\tau = Fr = mA\cos(\omega t - \alpha)[-2A\omega^2 \sin(\omega t - \alpha) + g\sin(\omega t)] .$$

(d) There is no loss of generality in putting $\alpha = 0$. Then

$$\tau = mA\left(\frac{g}{2} - A\omega^2\right)\sin(2\omega t) .$$

Hence $\tau \leq mA(\frac{g}{2} - A\omega^2)$ for the lower resonance. If the torque yielded by the driving clock spring is greater than this upper bound, ω will increase and the resonant state will no longer hold.

1087

A mass m_1 moves around a hole on a frictionless horizontal table. The mass is tied to a string which passes through the hole. A mass m_2 is tied to the other end of the string (Fig. 1.61).

(a) Given the initial position $\mathbf{R}_0$ and velocity $\mathbf{V}_0$ in the plane of the table and the masses m_1 and m_2, find the equation that determines the maximum and minimum radial distances of the orbit. (Do not bother to solve it!)

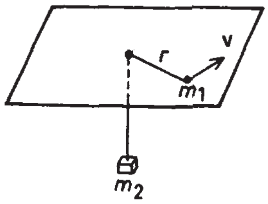

Fig. 1.61.

(b) Find the frequency of oscillation of the radius of the orbit when the orbit is only sightly different from circular.

(*Princeton*)

Solution:

(a) The equations of motion of m_1 and m_2 are

$$m_1(\ddot{r} - r\dot{\theta}^2) = -T \ , \tag{1}$$

$$m_1 r^2 \dot{\theta} = m_1 h \ , \tag{2}$$

$$T - m_2 g = m_2 \ddot{r} \ , \tag{3}$$

where $m_1 h$ is the angular momentum, a constant. Eliminating T from (1) and (3), we obtain

$$(m_1 + m_2)\ddot{r} - m_1 r\dot{\theta}^2 + m_2 g = 0 \ . \tag{4}$$

Equations (2) and (4) give

$$(m_1 + m_2)\ddot{r} - \frac{m_1 h^2}{r^3} = -m_2 g \ . \tag{5}$$

As $\ddot{r} = \dot{r}\frac{d\dot{r}}{dr} = \frac{1}{2}\frac{d\dot{r}^2}{dr}$, the above can be readily integrated to give

$$\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{m_1 h^2}{2r^2} = -m_2 g r + C \ . \tag{6}$$

At $t = 0$, $r = R_0$, $\dot{r} = V_0 \cos\phi$, $r\dot{\theta} = V_0 \sin\phi$, so that $h = R_0 V_0 \sin\phi$, where ϕ is the angle between $\mathbf{R}_0$ and $\mathbf{V}_0$. Then the constant of integration C can be evaluated as

$$C = \frac{1}{2}[(m_1 + m_2)V_0^2 \cos^2\phi + m_1 V_0^2 \sin^2\phi] + m_2 g R_0 \ .$$

For r to be an extremum, $\dot{r} = 0$, with which (6) becomes

$$2m_2gr^3 - 2Cr^2 + m_1h^2 = 0 \ ,$$

whose solution gives the maximum and minimum radial distances of r.

(b) When the orbit of m_1 is circular, $\ddot{r} = 0$, and (5) gives

$$h^2 = \frac{m_2gr_0^3}{m_1} \ , \tag{7}$$

where r_0 is the radius of the circular orbit. When the orbit is slightly different from circular, let $r = r_0 + x$, where $x \ll r_0$. Equation (5) then becomes

$$(m_1 + m_2)\ddot{x} - m_1h^2/(r_0 + x)^3 = -m_2g \ .$$

As

$$(r_0 + x)^{-3} = r_0^{-3}\left(1 + \frac{x}{r_0}\right)^{-3} \approx r_0^{-3}\left(1 - \frac{3x}{r_0}\right) \ ,$$

the above becomes

$$(m_1 + m_2)\ddot{x} - m_1h^2(r_0^{-3} - 3xr_0^{-4}) = -m_2g \ .$$

Then using (7) we have

$$(m_1 + m_2)\ddot{x} + \frac{3m_2gx}{r_0} = 0 \ .$$

This shows that x oscillates simple-harmonically with frequency

$$\frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3m_2g}{(m_1 + m_2)r_0}} \ .$$

1088

(a) Consider a damped, driven harmonic oscillator (in one dimension) with equation of motion

$$m\ddot{x} = -m\omega_0^2x - \gamma\dot{x} + A\cos(\omega t) \ .$$

What is the time-averaged rate of energy dissipation?

(b) Consider an anharmonic oscillator with equation of motion

$$m\ddot{x} = -m\omega_0^2x + \alpha x^2 + A\cos(\omega t) \ ,$$

where α is a small constant.